The Clothoid

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Abstract

In this paper we will discuss the properties of the special plane curve known as the Clothoid.

1 Introduction

The Clothoid, also known as Euler's Spiral or Spiral of Cornu, is a plane curve defined by the parametric equations

$$f(t) = \int_0^t sin\left(\frac{x^2}{2}\right) dx$$

$$g(t) = \int_0^t cos\left(\frac{x^2}{2}\right) dx$$

These equations are known as the Fresnel integrals, so-called because of their association with the phenomenon known as Fresnel diffraction, a branch of the study of optics. As early as 1743, Euler considered these integrals in connection with a problem set by Jacob Bernoulli in the late 1600s: "To find the curvature a lamina should have in order to be straightened out horizontally by a weight suspended at one end.[1]"

In the early nineteenth century, Augustin-Jean Fresnel studied these integrals in relation to his work on the diffraction of light, and later that century, Marie Alfred Cornu was the first to accurately plot the spiral. (Euler had only given half the spiral.)[1]

2 Properties of the Clothoid

The Clothoid has a number of interesting relationships involved in its physical properties. First, let's take a look at the arc length.

2.1 Arc Length

According to a theorem in Stewart's *Early Transcendentals*[2], the length s of a curve C, described by parametric equations $x = f(t)$ and $y = g(t)$, on the interval $\alpha \leq t \leq \beta$, where $f'(t)$ and $g'(t)$ are continuous on $[\alpha, \beta]$, is given by

$$s = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Using this theorem, and applying the Fundamental Theorem of Calculus, we compute the length of an arc of the Clothoid on the interval $[0, t]$:

$$\begin{aligned} s &= \int_0^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du \\ &= \int_0^t \sqrt{sin^2 u + cos^2 u}\, du \\ &= \int_0^t du \\ &= t \end{aligned}$$

Thus, we find that the length of an arc of the Clothoid is equal to its parameter, t. We can now show that, beginning with $\vec{\mathbf{r}}(t)$, the Clothoid can be parameterized as a function of its curvature, by showing that curvature is equal to arc length.

2.2 Curvature

Because $s(t) = t$, the reparametrization of $\vec{\mathbf{r}}$ in terms of s is given by:

$$\begin{aligned}
\vec{\mathbf{r}}(s) &= \left\langle \int_0^s sin\left(\frac{u^2}{2}\right) du, \int_0^s cos\left(\frac{u^2}{2}\right) du \right\rangle \\
\vec{\mathbf{r}}\,'(s) &= \left\langle sin\left(\frac{s^2}{2}\right), cos\left(\frac{s^2}{2}\right) \right\rangle \\
\hat{\mathbf{T}}(s) &= \frac{\vec{\mathbf{r}}\,'(s)}{\|\vec{\mathbf{r}}\,'(s)\|} \\
&= \vec{\mathbf{r}}\,'(s) \\
\hat{\mathbf{T}}'(s) &= \vec{\mathbf{r}}\,''(s) \\
&= \left\langle (s)cos\left(\frac{s^2}{2}\right), -(s)sin\left(\frac{s^2}{2}\right) \right\rangle \\
\kappa(s) &= \frac{\left\|\hat{\mathbf{T}}'(s)\right\|}{\|\vec{\mathbf{r}}\,'(s)\|} \\
&= \frac{\left\|\left\langle (s)cos\left(\frac{s^2}{2}\right), -(s)sin\left(\frac{s^2}{2}\right) \right\rangle\right\|}{\left\|\left\langle sin\left(\frac{s^2}{2}\right), cos\left(\frac{s^2}{2}\right) \right\rangle\right\|} \\
\kappa(s) &= s
\end{aligned}$$

This shows us that the Clothoid is a curve whose curvature at any point is equal to its arc length (measured from the origin). This explains the spiral's rapid degeneration as s approaches ∞.

Now, suppose we're given a curvature function κ in terms of arc length s, and we wish to obtain the parametric equations $f(t)$ and $g(t)$ for the curve which the function describes. Again, we will take a vector approach to show that such a curve can be defined by

$$\vec{\mathbf{r}}(s) = \left\langle \int_0^s sin\ \theta(u)\ du, \int_0^s cos\ \theta(u)\ du \right\rangle,$$

where $\theta(u) = \int_0^u \kappa(t)dt$.

First, it should be noted, $\kappa(s) = \left\|\frac{d\hat{\mathbf{T}}}{ds}\right\|$ and $\theta = \theta(s)$.

Proof.

$$\begin{aligned}
\vec{\mathbf{r}}(s) &= \left\langle \int_0^s sin\ \theta(u)\ du, \int_0^s cos\ \theta(u)\ du \right\rangle \\
\hat{\mathbf{T}}(s) &= \frac{d\vec{\mathbf{r}}}{ds} \\
&= \langle sin\ \theta(s), cos\ \theta(s) \rangle \\
\left\|\frac{d\hat{\mathbf{T}}}{ds}\right\| &= \left\|\left\langle cos\ \theta(s)\frac{d\theta}{ds}, -sin\ \theta(s)\frac{d\theta}{ds} \right\rangle\right\| \\
&= \left|\frac{d\theta}{ds}\right| \|\langle cos\ \theta(s), -sin\ \theta(s)\rangle\| \\
&= \left|\frac{d\theta}{ds}\right| \\
&= \kappa(s)
\end{aligned}$$

□

The above proof shows that, knowing only the curvature as a function of arc length, we can obtain the parametrization of a curve (in terms of arc length). Let's demonstrate this by parametrizing the unit circle.

3 Examples

3.1 Unit Circle

The curvature function of the unit circle is constant, specifically:

$$\kappa(s) = 1$$

Now, since $\kappa(s) = \theta'(s)$, we can integrate $\kappa(s)$ to obtain:

$$\theta(s) = s$$

Substituting this into our parametrization $\vec{\mathbf{r}}$ then integrating yields the following:

$$\begin{aligned} \vec{\mathbf{r}}(s) &= \left\langle \int_0^s sin\ \theta(u)\ du, \int_0^s cos\ \theta(u)\ du \right\rangle \\ &= \left\langle \int_0^s sin\ u\ du, \int_0^s cos\ u\ du \right\rangle \\ &= \langle -cos\ s, sin\ s \rangle \end{aligned}$$

We can observe that $\vec{\mathbf{r}}(s)$ now describes a circle of radius 1.

3.2 Clothoid

Next, let's take a look at some parametrizations using different curvature functions.

First, we know the curvature function of the Clothoid to be $\kappa(s) = s$. Integrating the curvature function gives us $\theta(s) = \frac{s^2}{2}$. We now observe that substituting this into our general parametrization gives us the equations for the Clothoid.

4 Curvature Revisited

The above proof shows us that a curve can be expressed parametrically in terms of arc length s using the general formula

References

[1] *Euler's Spiral* American Math Monthly, Volume 25 (1918)

[2] Stewart, James. *Early Transcendentals*, 5th ed. Brooks/Cole Belmont, California 2003.